Simply Logic and numbers

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Question

Let x and y be two integers satisfying **(i)** $x > y > 1$ and **(ii)** $x + y \leq 100$

Assume that Mr. Sum knows $x + y$ and Mr. Product knows xy.

(At the same time they know that their partner obtains such information.)

Mr. Product :	I cannot determine the values of x and y.	**(1)**
Mr. Sum :	I know earlier that you cannot determine that.	**(2)**
Mr. Product :	Ha! Then I know the values of x and y.	**(3)**
Mr. Sum :	Hurray! I also know their values.	**(4)**

What are the values of x and y ?

Solution

From **(1)** , we know that x and y cannot both are prime numbers at the same time. If x and y are both prime, then Mr. Product knew the values of x and y . For example, if $xy = 6$, then $x = 3$ and $y = 2$. ($x > y > 1$).

Since all even numbers greater than 2 can be decomposed into the sum of 2 prime number[1], then $x + y$ cannot be even or a prime number plus 2. Otherwise, from **(2)** , Mr. Sum could not say that "I know **earlier** …"

From the conditions **(i)** $x > y > 1$ and **(ii)** $x + y \leq 100$, the remaining values of $x + y$ are :

11, 17, 23, 27, 29, 35, 37, 41, 47, 51, 53, 57, 59, 65, 67, 71, 77, 79, 83, 87, 89, 93, 95, 97 ***(List 1)***

If $x + y = 57$, then there is a possibility that $x = 53$, $y = 4$, $xy = 212$. Then Mr. Product would know that there are two possible cases for x and y :

(a) $x = 53, y = 4$

(b) $x = 106, y = 2$

But from **(ii)** $x + y \leq 100$, the only answer must be case **(a)** . Then Mr. Product "**may be possible**" to know the values of x and y . This contradicts to what Mr. Sum said in **(2)** : "I know **earlier** that you cannot determine that." These lines of thinking apply to all other values greater than 57 . Therefore $x + y < 57$.

Since $x + y < 57$, ***(List 1)*** for the value of $x + y$ above becomes:

11, 17, 23, 27, 29, 35, 37, 41, 47, 51, 53 ***(List 2)***

From **(2)** , Mr. Product knew that $x + y$ cannot be even or a prime number plus 2. He then could find the values of x and y and said **(3)** . From here we visualize that:

By breaking xy into two factors (not necessarily both are prime) and rewrite in the form of $x + y$, there are usually many methods. However, there must be a method in which $x + y$ is odd and at the same time $x + y$ subtract 2 must not be a prime number. Since $x + y$ is odd, then one of the x and y is odd and the other is even. Then xy must be even. Finally we get $xy = 2^n p$, where p is odd.

We also know that:

(a) p is prime. Proof: If not, assume $p = p_1 p_2$ (may be more factors), $xy = 2^n p_1 p_2$,
so there are two possibilities for x and y:

(i) $x = 2^n p_1$, $y = p_2$;

(ii) $x = 2^n$, $y = p_1 p_2$,

If $p_1 \neq p_2$, we have the third possibility:

(iii) $x = 2^n p_2$, $y = p_1$.

As a result, Mr. Product could not determine the values of x and y and said **(3)** .

Thus, $xy = 2^n p$, where p is odd prime number.

(b) n = 2, 3, 4, 5. Proof : If $n = 6$ (or above), then $x + y = 2^6 + p = 64 + p > 57$, this contradicts to the fact that $x + y < 57$ which we proved above. When $n = 1$, $xy = 2p$. Then $x = p$ and $y = 2$, and Mr. Product could not say **(1)** .

Up to now, $xy = 2^n p$, then $x + y = 2^n + p$ (Note that one of x and y is odd, and the other is even.) Therefore,

$(x + y) - 2^n = p$, where p is prime.

Now we deduct from ***(List 2)*** the values 4, 8, 16, 32 (that is 2^n, where n = 2, 3, 4, 5). If there is “only one of” such differences is a prime number, then this may be the possible value for $(x + y)$. However, if there is “more than one of” differences are prime numbers, then Mr. Product could not tell the values of x and y and said **(3)** .

For example, 17 – 4 = 13 ; 17 – 8 = 9 ; 17 – 16 = 1.

Since 9 and 1 are not prime numbers, 13 is the only prime number and 17 may be the number we want.

But 23 – 4 = 19, 23 – 8 = 15, 23 – 16 = 17.

Since 19 and 17 are prime numbers, 23 is not the number we want.

After calculating and deducting from ***(List 2)*** we get the possible values of $x + y$:

17, 29, 41, 53 ***(List 3)***

If $x + y = 29$, let us suppose $x = 18$ and $y = 11$, $xy = 198$. Then Mr. Product may think of several values for $x + y$:

(a) 2 + 99 = 101 (Note that 2 x 99 = 198)

(b) 3 + 66 = 69

(c) $6 + 33 = 39$

(d) $9 + 22 = 31$

(e) $11 + 18 = 29$

Since $x + y \le 100$, **(a)** does not establish.

Since $x + y - 2$ must not be prime, therefore **(b)**, **(c)**, **(d)** do not establish.

As a result Mr. Product knew that **(e)** is good, that is, $x = 18$ and $y = 11$.

However, $x + y = 29 = 18 + 11$, it can also be written as

$x + y = 16 + 13$, where 16 is written in 2^n form, and 13 is prime.

So, although Mr. Product could tell the values of x and y, Mr. Sum could not and could not said **(4)**. Therefore $x + y \neq 29$.

With the same argument, $x + y \neq 41$ or 53. The only possible value left behind is

$x + y = 17 = 13 + 4$, where 4 is in 2^n form, and 13 is prime.

The final solution : $x = 13$ and $y = 4$.

We can check our answer:

(a) $x + y = 17$, $xy = 52$, $x = 13$, $y = 4$ satisfy the conditions :

(i) $x > y > 1$ and **(ii)** $x + y \le 100$.

(b) $xy = 52 = 2 \times 26 = 4 \times 13$, so Mr. Product could not fix the values of x and y and said sentence **(1)**.

(c) $x + y = 17 = 15 + 2 = 14 + 3 = 13 + 4 = 12 + 5 = 11 + 6 = 10 + 7 = 9 + 8$, there is no set of numbers which are both prime numbers. So Mr. Sum knows Mr. Product could not fix the values of x and y and said sentence **(2)**.

(d) According to what Mr. Sum said in sentence **(2)**, Mr. product knew that $x + y$ is odd, there is only one possible way : $x = 13$ and $y = 4$, so he said sentence **(3)**.

(e) Hearing what Mr. Product said, Mr. Sum knew that $xy = 2^n p$, and $x + y = 2^n + p$. Since there is only one method to decompose 17 into the form $2^n + p$, that is, $17 = 2^2 + 13$, so Mr. Sum knew that $x = 13$ and $y = 4$ and he said sentence **(4)**.

Reference and others

1. The Goldbach Conjecture is a well-known unsolved problem. It states that any even number, which is greater than two, can be expressed as the sum of two primes. However, for small numbers in this article, the conjecture is sound. Further reading: http://en.wikipedia.org/wiki/Goldbach's_conjecture .
2. The original source of this question is unknown. The author got it from the orientation camp of the Chinese University in 1974.
3. The author would like to thank Ms. Chan Lap Lin, H.O.D. of mathematics, Queen’s College for her careful proof-reading.